2nd Order transfer function - Summary of results

The canonical 2nd order transfer function is expressed as

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]  

\( \omega_n \) is the **natural frequency**; \( \zeta \) is the **damping** coefficient.

The Poles, \( s_1, s_2 \) are

\[ s_1, s_2 = \omega_n \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) \]

On the right is the root locus for fixed \( \omega_n \) and varying \( \zeta \).

The transfer function in the frequency domain is

\[ H(\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + 2\zeta\omega_n \omega} = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2}} e^{i\Phi} \]

with \( \tan\Phi = -\frac{2\zeta\omega_n \omega}{\omega_n^2 - \omega^2} \)

The Bode plot is:

For \( \zeta = \frac{1}{\sqrt{2}} \) we have the sharpest corner without a maximum.

For \( \zeta < \frac{1}{\sqrt{2}} \) there is a peak in the plot of magnitude

\[ |H|_{\text{max}} = \frac{1}{2\zeta \sqrt{1-\zeta^2}} \]

This is above the level at \( \omega = 0 \).

For the \( H(s) \) given at \( t=0 \) then \( |H| = 1 \)

The **roll-off rate** at high frequencies is 40 dB per decade (in frequency).

The overall phase change is -180°.

For low \( \zeta \) the swing in phase is all close to the frequency at the maximum \( |H| \).

The peak itself is at

\[ \omega_p = \omega_n \sqrt{1 - 2\zeta^2} \]

Beware; later we introduce

\[ \omega_d = \omega \sqrt{1 - \zeta^2} \]
Impulse Responses

$\zeta = 0$ Undamped

\[ y(t) = \omega_n \sin(\omega_n t) \]

$\zeta > 0$ Damped

\[ y(t) = \frac{1}{2\omega_n \sqrt{\zeta^2 - 1}} \left( \exp(s_1 t) - \exp(s_2 t) \right) \]

$\zeta = 1$ Critically damped

\[ y(t) = \omega_n^2 t \exp(-\omega_n t) \]

Curve is very similar to damped response

$\zeta < 0$ Underdamped

\[ y(t) = \frac{\omega_n^2}{\omega_d} \exp(-\zeta \omega_n t) \sin(\omega_d t) \]

The logarithm of the ratio of successive peaks is the logarithmic decrement or log-dec for short

\[ \delta = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}} \approx 2\pi \zeta \text{ if } \zeta << 1 \]
**Step responses**

\( \zeta = 0 \) Undamped

\[ y(t) = \left( 1 - \cos(\omega_n t) \right) \]

\( \zeta > 0 \) Damped

\[ y(t) = \left( 1 + \frac{1}{\sqrt{\zeta^2 - 1}} \left[ s_2 \exp(s_1 t) - s_1 \exp(s_2 t) \right] \right) \]

\( \zeta = 1 \) Critically damped

\[ y(t) = \left( 1 - \exp(-\omega_n t) \left[ 1 + \omega_n t \right] \right) \]

\( \zeta < 0 \) Underdamped

First peak*, \( M_P = 1 + \exp\left( -\frac{\pi \zeta}{\sqrt{1-\zeta^2}} \right) \)

Proportional overshoot:

\[ O_P = y - 1 = \exp\left( -\frac{\pi \zeta}{\sqrt{1-\zeta^2}} \right) \]

Time to be within \( \delta \), \( T_s = -\frac{\ln(\delta)}{\zeta \omega_n} \)

For \( \delta = 0.02 \), i.e., 2% : \( T_s \approx \frac{4}{\zeta \omega_n} \)

Root Total square error = \( \sqrt{\frac{1 + 4\zeta^2}{4\zeta \omega_n}} \)

* Some authors call \( M_P \) the overshoot.

**Frequency of oscillation:**

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]

**Period of oscillation:**

\[ T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} \]

For least root square total deviation \( \zeta = 1/2 \).

Then \( O_P \approx 16\% \)
Ramp response  Now set  \( f(t) = t \) as illustrated.

\[
F = \frac{1}{s^2}
\]

Perhaps unkindly I leave derivations to the reader. The general response is:

\[
Y = H(s) \times \frac{1}{s^2} = \frac{\omega_n^2}{s^2 (s^2 + 2\zeta \omega_n s + \omega_n^2)}
\]

\( \zeta = 0 \) Undamped

\[
Y = \left( \frac{1}{s^2} - \frac{1}{s^2 + \omega_n^2} \right)
\]

\[
y = \left( t - \frac{\sin(\omega_n t)}{\omega_n} \right)
\]

Plot is for \( \omega_n = 1 \); \( y = t - \sin(t) \)

Note: \( s_1 s_2 = \omega_n^2 \); \( s_1 + s_2 = -2\zeta \omega_n \)

\( \zeta > 0 \) Damped: \( s_1, s_2 = \omega_n \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) \)

\[
Y = \left[ \frac{1}{s^2} + \frac{s_1+s_2}{s\omega_n^2} + \frac{1}{s_1-s_2} \left( \frac{s_2}{s-s_1} - \frac{s_1}{s-s_2} \right) \right]
\]

\[
y = \left[ t - \frac{2\zeta}{\omega_n} + \frac{1}{s_1-s_2} (s_2 e^{s_1 t} - s_1 e^{s_2 t}) \right]
\]

Plot is with \( \omega_n = 1, \zeta = 2 \)

\( \zeta = 1 \) Critically Damped

\[
Y = \frac{1}{\omega_n} \left( \frac{\omega_n}{s^2} - \frac{2}{s + \omega_n} + \frac{\omega_n}{(s + \omega_n)^2} \right)
\]

\[
y = \frac{1}{\omega_n} \left( \omega_n t - 2 + (2 + \omega_n t) e^{-\omega_n t} \right)
\]

The plot is for \( \omega_n = 1 \).
\( \zeta < 0 \text{ Underdamped: } \omega_d = \omega_n \sqrt{1 - \zeta^2} \)

\[
Y = \frac{1}{\omega_n} \left( \frac{\omega_n}{s^2 - \frac{2\zeta}{s} + \frac{2\zeta(s + \zeta\omega_n)}{(s + \zeta\omega_n)^2 s + \omega_n^2(1 - \zeta^2)}} \right)
\]

\[
y = \frac{1}{\omega_n} \times \left( \omega_n t - 2\zeta e^{-\omega_n \zeta t} [2\zeta \cos(\omega_d t) - \omega_n (1 - 2\zeta^2) \sin(\omega_d t)] \right)
\]

The plot is with \( \omega_n = 1 \) and \( \zeta = 0.2 \)

**Second order transfer function analysis**

As an example consider the mass, damper and spring acted on by a force, \( f(t) \). We had for the displacement, \( y \) :-

\[
m \ddot{y} + c \dot{y} + k y = f
\]

Take the Laplace transform (with no initial values) :-

\[
m s^2 Y + c s Y + k Y = F
\]

or

\[
s^2 Y + \frac{c}{m} s Y + \frac{k}{m} Y = F/m
\]

which with \( \omega_n = \sqrt{\frac{k}{m}} \) and \( \zeta = \frac{c}{2\sqrt{km}} \) becomes If we redefine \( F \equiv \frac{F}{\omega_n^2 m} = \frac{F}{k} :-

\[
s^2 Y + 2\zeta \omega_n s Y + \omega_n^2 = F \omega_n^2 : \text{ or } (s^2 + 2\zeta \omega_n s + \omega_n^2) Y = F \omega_n^2
\]

and so

\[
H(s) = \frac{F}{Y} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \quad \text{...1}
\]

This equation 1 is a standard form for any second order transfer function. Whatever the physical variables it helps to turn the expression to this format early in the analysis.

\( \omega_n \) is the natural frequency; \( \zeta \) is the damping coefficient. \quad \text{...2}

**Poles**

Poles, \( s_1, s_2 \) are the values of \( s \) for which the denominator is zero.

\[
s_{1,2} = \frac{-2\zeta \omega_n \pm \sqrt{4 \zeta^2 \omega_n^2 - 4 \omega_n^2}}{2} = \omega_n \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) \quad \text{...3}
\]
Cases:

1. \( \zeta = 0 \): poles at \( s = \pm j\omega_n \).

2. \( \zeta < 1 \): two complex poles. These lie on a circle of radius \( \omega_n \).

3. \( \zeta = 1 \): two coincident poles at \( s = -\omega_n \).

4. \( \zeta > 1 \): two poles on the negative real axis.

These are illustrated on the root locus diagram on the right for constant \( \omega_n \).
Bode Plot

From eq. 1

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

and putting \( s \to j\omega \)

\[ H(\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j 2\zeta \omega_n \omega} \]  …4a

\[ = |H(\omega)| \times \exp(j\Phi) \]  …4b

with \[ |H(\omega)| = \frac{\omega_n^2}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + 4 \zeta^2 \omega_n^2 \omega^2}} \]  …4c

and \[ \tan\Phi = -\frac{2\zeta \omega_n \omega}{\omega_n^2 - \omega^2} \]  …4d

\[
\begin{array}{|c|c|c|}
\hline
& \omega \to 0 & \omega = \omega_n & \omega \to \infty \\
\hline
|H| & 1 & \frac{1}{2\zeta} & \frac{\omega_n^2}{\omega^2} \\
\Phi & 0 & -\frac{\pi}{2} & -\pi \\
\hline
\end{array}
\]

For \( \zeta < \frac{1}{\sqrt{2}} \) there is a peak in the \(|H|\) curve. It is easier to find the peak of \(|H|^2\) which is at the same value of \( \omega \) as the peak of \(|H|\). And easier still to find the minimum of \(|H|^2\) which occurs at the same value of \( \omega \) as the peak in \(|H|\)

\[ \omega_n^4 \frac{d}{d\omega} \left( \frac{1}{|H|^2} \right) = \frac{d}{d\omega} \left( (\omega_n^2 - \omega^2)^2 + 4 \zeta^2 \omega_n^2 \omega^2 \right) = 2(\omega_n^2 - \omega^2)(-2\omega) + 8\zeta^2 \omega_n^2 \omega \]

\[ = 4\omega \left( \omega^2 - \omega_n^2 \left( 1-2\zeta^2 \right) \right) \]

so \[ \frac{d}{d\omega} \left( \frac{1}{|H|^2} \right) = 0 \]  if \( \omega = 0; \) or if \( \omega = \omega_n \sqrt{1-2\zeta^2} \)

In the latter case \[ |H| = \frac{1}{\sqrt{\left( 1-1+2\zeta^2 \right)^2 + 4 \zeta^2 (1-2\zeta^2)}} \]

\[ = \frac{1}{\sqrt{4 \zeta^2 (1-\zeta^2)}} = \frac{1}{2\zeta \sqrt{1-\zeta^2}} \]  …5

Note that if \( \zeta = 0 \) then the peak of \(|H| \to \infty \). This is the undamped case. Note also that the peak is at a frequency close to but slightly less than \( \omega_n \). There is no peak if \( \zeta > \frac{1}{\sqrt{2}} \) since then the frequency is imaginary.

This is illustrated in the next figures with \( \omega_n = 1 \). Note that the frequency scale has been extended in the phase plot to show the approach to the limits.

In the magnitude plot note (i) the roll of rate is 40 dB per decade.
(ii) the peak is at about 14 dB for $\zeta = 0.1$. Eq. 5 gives a peak of a factor of 5: so $20 \log_{10}(5) = 13.97$. The theory pans out correctly.

**Impulse Responses**

Consider the Dirac impulse, $\delta(t)$, as an input.

$$\zeta = 0, \quad Y = \frac{\omega_n^2}{s^2 + \omega_n^2} \times 1 = \frac{\omega_n \times \omega_n}{s^2 + \omega_n^2}$$

so $y(t) = \omega_n \sin(\omega_n t)$

There is no damping and the system rings forever.
\section*{Second order transfer functions}

\subsection*{\(\zeta > 1\)}

The roots are real. We had eq. 3:

\[ s_1, s_2 = \omega_n \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) ; s_1 > s_2 \quad \ldots 3 \]

Thus the response is

\[ Y = \frac{\omega_n^2}{(s-s_1)(s-s_2)} \times 1 \]

\[ = \frac{\omega_n^2}{s_1-s_2} \left( \frac{1}{s-s_1} - \frac{1}{s-s_2} \right) \]

and

\[ y(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left( \exp(s_1 t) - \exp(s_2 t) \right) \]

The response below is for \(\omega_n = 1\) and \(\zeta = 2\). The coefficients of the component exponentials is omitted in the figure. The formula is:

\[ y = 0.289( e^{-0.27 t} - e^{-3.7 t} ) \]

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{plot.png}
\caption{Response plot for \(\omega_n = 1\) and \(\zeta = 2\).}
\end{figure}

\subsection*{\(\zeta = 1\)}

\[ Y = \frac{\omega_n^2}{(s + \omega_n)^2} \times 1 = \frac{\omega_n^2}{(s + \omega_n)^2} \]

so

\[ y(t) = \omega_n^2 t \exp(-\omega_n t) \]

The plot is very like the previous one.

\subsection*{\(\zeta < 1\)}

This is a bit more complicated. We need to complete squares in the denominator.

\[ Y = H(s) \times 1 = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

\[ = \frac{\omega_n^2}{(s + \zeta \omega_n)^2 - \zeta^2 \omega_n^2 + \omega_n^2} \]

\[ = \frac{\omega_n^2}{(s + \zeta \omega_n)^2 + \omega_d^2} \quad \ldots 6 \]

with

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \quad \ldots 7 \]

This frequency appears often and should be noted.
Thus\[ Y = \frac{\omega_n^2}{\omega_d} \times \frac{\omega_d}{(s + \zeta \omega_n)^2 + \omega_d^2} \]

so, from the tables\[ y(t) = \frac{\omega_n^2}{\omega_d} \exp(-\zeta \omega_n t) \sin(\omega_d t) \quad \ldots 8a \]
\[ = \frac{\omega_n}{\sqrt{1-\zeta^2}} \exp(-\zeta \omega_n t) \sin(\omega_d t) \quad \ldots 8b \]

The response for $\omega_n = 1$ and $\zeta = 0.1$ is below. It is an exponentially decaying sine wave.

The formula is: \[ y = 1.005 e^{-0.1 t} \sin(0.995 t) \]

The period of the oscillation, $T = \frac{2\pi}{\omega_d}$. The ratio of succeeding positive peaks is then:

\[ \frac{\exp(-\zeta \omega_n t)}{\exp(-\zeta \omega_n \left( t + \frac{2\pi}{\omega_d} \right))} = \exp \left( \frac{2\pi \zeta \omega_n}{\omega_d} \right) \]

The logarithm of this ratio is

\[ \delta = \frac{2\pi \zeta \omega_n}{\omega_d} = \frac{2\pi \zeta \omega_n}{\omega_n \sqrt{1-\zeta^2}} \]

so \[ \delta = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}} \approx 2\pi \zeta \text{ if } \zeta \ll 1 \quad \ldots 9 \]

$\delta$ is called the logarithmic decrement or log-dec for short. It is a useful way of measuring the damping in a highly resonant system.

Note we use the symbol $\delta$ with another meaning later.
Step function responses

The input is \( f = u(t) \)

so

\[ F = \frac{1}{s} \]

The response is, using \( H(s) \) from eq. 1

\[ Y = \frac{\frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}}{s} \times \frac{1}{s} \]

...10

Cases

\( \zeta = 0 \)

\[ Y = \frac{\frac{\omega_n^2}{s^2 + \omega_n^2}}{s} \equiv \frac{A}{s} + \frac{Bs + C}{s^2 + \omega_n^2} \]

Thus

1 \( \equiv \) \( A(s^2 + \omega_n^2) + s(Bs + C) \)

Put \( s = 0 \):

\( \omega_n^2 = A \omega_n^2 \) \( \therefore \) \( A = 1 \)

Coeff. \( s^2 \):

0 \( = A + B \) \( \therefore B = -A = -1 \)

Coeff. \( s \):

0 \( = C \) \( \therefore C = 0 \)

\[ Y = \left( \frac{1}{s} - \frac{s}{s^2 + \omega_n^2} \right) \]

and \( y(t) = \left( 1 - \cos(\omega_n t) \right) \)

The plot for \( \omega_n = 1 \) is on the right.

The system oscillates about the mean indefinitely.

\( \zeta > 1 \)

The transfer function is factorisable and the roots are real. As before eq. 3 gives the roots:

\[ s_1, s_2 = \omega_n \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) ; s_1 > s_2 \]

\( \therefore \)

...3

and the response is

\[ Y = \frac{\omega_n^2}{(s-s_1)(s-s_2)} \times \frac{1}{s} \]

\[ \therefore \]

We have

\[ Y = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} \equiv \frac{A}{s} + \frac{B}{s-s_1} + \frac{B}{s-s_2} \]
Second order transfer functions

\[
1 = A(s - s_1)(s - s_2) + Bs(s - s_2) + Cs(s - s_1)
\]

Put \(s = s_1:\)
\[
\omega_n^2 = B s_1(s_1 - s_2) \quad \therefore \quad B = \frac{\omega_n^2 s_1(s_1 - s_2)}{s_1 - s_2}
\]

Put \(s = s_2:\)
\[
\omega_n^2 = C s_2(s_2 - s_1) \quad \therefore \quad C = -\frac{\omega_n^2 s_2(s_2 - s_1)}{s_1 - s_2}
\]

Put \(s = 0:\)
\[
\omega_n^2 = A s_1 s_2 \quad \therefore \quad A = \frac{\omega_n^2}{s_1 s_2}
\]

So
\[
Y = \frac{1}{s} + \frac{1}{\omega_n \sqrt{\zeta^2 - 1}} \left( \frac{s_2}{s - s_1} - \frac{s_1}{s - s_2} \right)
\]

and
\[
y(t) = 1 + \frac{1}{\omega_n \sqrt{\zeta^2 - 1}} [s_2 \exp(s_1 t) - s_1 \exp(s_2 t)]
\]

noting \(s_1, s_2 = \omega_n \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) ; s_1 > s_2\)

With \(\omega_n = 1\) and \(\zeta = 2\) this becomes
\[
y(t) = 1 + 0.077 e^{-3.73 t} - 1.077 e^{-0.27 t}
\]

which is illustrated on the right.

\[
\zeta = 1
\]
\[
Y = \frac{\omega_n^2}{(s + \omega_n)^2} \times \frac{1}{s} = \frac{\omega_n^2}{s(s + \omega_n)^2}
\]

Thus
\[
\omega_n^2 = A(s + \omega_n)^2 + Bs(s + \omega_n) + Cs
\]

Put \(s = 0:\) \(\omega_n^2 = A\omega_n^2\) \(\therefore\) \(A = 1\). Put \(s = -\omega_n:\) \(\omega_n^2 = -C\omega_n\) \(\therefore\) \(C = -\omega_n\)

Coeff. \(s^2:\)
\[
0 = A + B \quad \therefore \quad B = -A = -1
\]

\[
Y = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}
\]

Thus from the Tables:
\[
y(t) = 1 - \exp(-\omega_n t) \left[1 + \omega_n t\right]
\]

For \(\omega_n = 1\) \(y(t) = 1 - e^{-t} \left(1 + t\right)\)

This is illustrated on the right.

\(\zeta < 1\) The more complicated but more interesting case. From eq. 1 we have
\[
Y = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \times \frac{1}{s}
\]
\[
\frac{\omega_n^2}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta \omega_n s + \omega_n^2}, \text{ say.}
\]

Thus
\[
1 \equiv A (s^2 + 2\zeta \omega_n s + \omega_n^2) + s (B s + C)
\]

Put \(s = 0\):
\[
\omega_n^2 = A \omega_n^2 \quad \therefore \quad A = 1
\]

Coeff. \(s^2\): \(0 = A + B \quad \therefore \quad B = -A = -1\)

Coeff. \(s\): \(0 = A 2 \zeta \omega_n + C \quad \therefore \quad C = -2\zeta \omega_n A = -2\zeta \omega_n
\]

and
\[
Y = \frac{1}{s} - \frac{s + 2\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} : \text{completing the square as in eq. 6}
\]

\[
= \frac{1}{s} - \frac{(s + \zeta \omega_n) + \frac{\omega_n}{\omega_d} \omega_d}{(s + \zeta \omega_n)^2 + \omega_d^2} : \text{making the numerator to suit the Tables}
\]

with \(\omega_d = \omega_n \sqrt{1 - \zeta^2}\)

Thus
\[
y(t) = 1 - \exp(-\zeta \omega_n t) \left( \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \sin(\omega_d t) \right) \quad \ldots 11
\]

For \(\omega_n = 1\) and \(\zeta = 0.2\)

we have
\[
y(t) \approx 1 - e^{-0.2t} \left[ \cos(0.98 t) + 0.2 \sin(0.98 t) \right]
\]

This is plotted on the right.

Note that the gradient is zero at \(t = 0\)
There are properties of interest of the curve.

Overshoot

The derivative

\[
\frac{dy}{dt} = \frac{\omega_n^2}{\omega_d} \exp(-\zeta \omega_d t) \sin(\omega_d t)
\]

is remarkably simple. It is equal to the impulse response. Indeed it is generally the case that the derivative of the step response of any linear system is the impulse response.

\[
\frac{dy}{dt} = 0 \text{ when } \omega_d t = n\pi \text{ or when } t = \frac{n\pi}{\omega_d} = \frac{n\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{nT}{2}
\]

where the period of the oscillations is

\[
T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}}
\]

The value of \(y\) at these extrema is

\[
Y = 1 + \exp\left(-\frac{n\pi \zeta}{\sqrt{1-\zeta^2}}\right)
\]

The first of these when \(n = 1\) gives the peak of the overshoot (often simply called the overshoot).

\[
M_p = 1 + \exp\left(-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}\right)
\]

Express proportional overshoot as the excess as a proportion of the final steady state value.

\[
O_p = y - 1 = \exp\left(-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}\right) \quad \ldots 12a
\]

occurring at

\[
T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} = \frac{T}{2} \quad \ldots 12b
\]
We may be interested in the time to reach and stay within a certain 'distance' from the final value. The deviation of the nth extremum is

$$\delta = \exp\left(-\frac{n \pi \zeta}{\sqrt{1-\zeta^2}}\right)$$

which occurs at

$$t_n = \frac{n\pi}{\omega_n \sqrt{1-\zeta^2}}$$

or

$$\frac{n\pi}{\sqrt{1-\zeta^2}} = t_n \omega_n$$

so

$$\delta = \exp\left(-\frac{n \pi \zeta}{\sqrt{1-\zeta^2}}\right) = \exp(-\omega_n \zeta t_n)$$

and

$$T_s = t_n = -\ln(\delta) \frac{1}{\omega_n \zeta}$$

Strictly \(t_n\) should be an integral number of half periods but the discrepancy is usually ignored.

For \(\delta = 0.02\), i.e., 2% \(T_s \approx \frac{4}{\zeta \omega_n}\)

and for \(\delta = 0.05\), i.e., 5% \(T_s \approx \frac{3}{\zeta \omega_n}\)

We may also consider an overall expression of deviation from the final value. The root mean square seems a suitable choice.

The deviation and its square are illustrated above for the previously shown response. We take then as a measure of the overall deviation as

$$E_{rts} = \text{the total square deviation (integrated from } t = 0 \text{ to } t = \infty)$$

We require

$$E_{rts} = \sqrt{\int_0^\infty (y-1)^2 \, dt}^{1/2}$$

$$= \sqrt{\frac{1+4 \zeta^2}{4\omega_n \zeta}}$$

This rather simple expression is valid for all \(\zeta\), i.e., \(0 < \zeta < \infty\). It has a minimum value of 1 when \(\zeta = \frac{1}{2}\).
This minimum square deviation condition is illustrated on the right with a 16% overshoot.
There are other measures of the error. To put emphasis on the approach to the final value on might weight erros by muptiplying by time. The integral of this de-emphasises the initial approach to the final value. Thus we have

The root of the integral of time weighted total square erroe is

\[
E_{\text{rts}} = \left[ \int_{0}^{\infty} t (y - 1)^2 \, dt \right]^{1/2}
\]

\[
= \frac{1}{2\omega_n \zeta} \sqrt{1 + \frac{8}{\zeta^4}}
\]

This has a minimum of \( \frac{1}{2^{1/4} \omega_n} \) when

\[
\zeta = \left( \frac{1}{8} \right)^{1/4} \approx 0.595
\]

The overshoot is

\[
\exp\left( \zeta \pi \sqrt{1 - \zeta^2} \right) = \exp\left( \frac{\pi}{\sqrt{2\sqrt{2} - 1}} \right) = 9.8%\]
Keeping factors constant

Poles are at \( s = \omega_n \left( -\zeta \pm \sqrt{1-\zeta^2} \right) \):

Constant imaginary part.
\[
\omega_d = \omega_n \sqrt{1-\zeta^2} = \text{constant.}
\]
So Constant frequency in response

Constant real part:
\[
\omega_n \zeta \text{ is constant.}
\]
Fixed envelope
\[
y = 1 - \exp(-\zeta \omega_n t)
\]
and so time to get within a given limit is determined by \( \omega_n \zeta \).
E.g., \( T_{\delta=2\%} = \frac{4}{\zeta \omega_n} \)

Constant damping ratio, \( \zeta \)
Thus also constant overshoot:
\[
\text{Op} = \exp \left( -\frac{\pi \zeta}{\sqrt{1-\zeta^2}} \right)
\]
Note: \( \zeta = \sin \theta \) where \( \theta \) is marked in the figure.

There is scarcely any overshoot when \( \theta > 60^\circ \).